22/11/23

MATH2050A Tutorial

Midterm 2 Problems

Q4: Sps. $\{x_n\}$ is a sequence of positive real numbers s.t. $\left\{\frac{x_{n+1}}{x_n}\right\}$ is bounded.

1) Show that $\{x_n^{\frac{1}{n}}\}$ is bounded

2) Show $\limsup_{n\to\infty} x_n^{\frac{1}{n}} \leq \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}$

3) Show that $\leq$ cannot be improved to $=$ by giving an example of a strict inequality.

Pf: (1): Since $\frac{x_{n+1}}{x_n}$ is bounded, $\exists$ $m, M \in \mathbb{R}$ s.t. $m \leq \frac{x_{n+1}}{x_n} \leq M$ for all n.

since each $x_n > 0$

$$x_{n+1} = x_1 \cdot \frac{x_2}{x_1} \cdot \frac{x_3}{x_2} \cdot \dots \cdot \frac{x_{n+1}}{x_n} = \left(\prod_{k=1}^{n} \frac{x_{k+1}}{x_k}\right) x_1 \leq M^n x_1$$

So $x_{n+1}^{\frac{1}{n+1}} \leq M^{\frac{n}{n+1}} x_1^{\frac{1}{n+1}}$

Case 1: if $x_1 \leq M$, then $x_{n+1}^{\frac{1}{n+1}} \leq M^{\frac{n}{n+1}} x_1^{\frac{1}{n+1}} \leq M$

$M < x_1$, then $x_{n+1}^{\frac{1}{n+1}} \leq M^{\frac{n}{n+1}} x_1^{\frac{1}{n+1}} \leq x_1$

So $x_n^{\frac{1}{n}} \leq \max\{M, x_1\}$.

$$x_{n+1} = \left(\prod_{k=1}^{n} \frac{x_{k+1}}{x_k}\right) x_1 \geq m^n x_1 \implies x_{n+1}^{\frac{1}{n+1}} \geq m^{\frac{n}{n+1}} x_1 \overset{\text{2 cases}}{\implies} x_n^{\frac{1}{n}} \geq \min\{m, x_1\}.$$

So $x_n^{\frac{1}{n}}$ is bounded.

2) Let $B := \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}$. Then for all $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. if $n \geq N$, $\frac{x_{n+1}}{x_n} \leq B + \varepsilon$ $(*)$.

Then $$\frac{x_n}{x_N} = \frac{x_{N+1}}{x_N} \cdot \frac{x_{N+2}}{x_{N+1}} \cdots\cdots \frac{x_n}{x_{n-1}} = \left(\prod_{k=N}^{n-1} \frac{x_{k+1}}{x_k}\right) \leq (B+\varepsilon)^{n-1-N}.$$

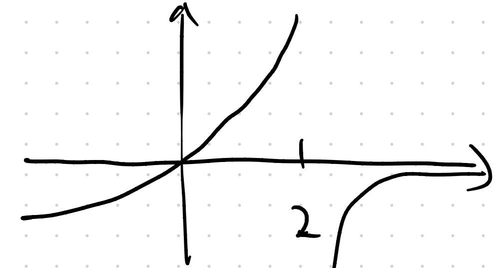
Then $$x_n \leq (B+\varepsilon)^{n-1-N} x_N = \frac{(B+\varepsilon)^n}{(B+\varepsilon)^{1+N}} x_N, \text{ so } x_n^{\frac{1}{n}} \leq \left(\frac{x_N}{(B+\varepsilon)^{1+N}}\right)^{\frac{1}{n}} (B+\varepsilon).$$

Note that for all $a > 0$, $a^{\frac{1}{n}} \to 1$ as $n \to \infty$.

So taking $n \to \infty$, $\limsup_{n\to\infty} x_n^{\frac{1}{n}} \leq B + \varepsilon$. Since this is for $\varepsilon > 0$ arbitrary, we obtain $\limsup_{n\to\infty} x_n^{\frac{1}{n}} \leq \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}$.

3) $x_n = \begin{cases} 1 & n \text{ odd} \\ 2 & n \text{ even} \end{cases}$. Then $\limsup_{n\to\infty} x_n^{\frac{1}{n}} = 1$, but $\frac{x_{n+1}}{x_n} = \left\{\frac{1}{2}, 2\right\}$, so $\limsup_{n\to\infty} \frac{x_{n+1}}{x_n} = 2$.

Q3: a) $\lim\limits_{x \to 0} \frac{x+2}{x^3-2} = 1$. Let $\varepsilon > 0$ be given.



Pf: $\left| \frac{x+2}{x^3-2} + 1 \right| = \left| \frac{x + x^3}{x^3-2} \right| = \frac{|x|\,|1+x^2|}{|x^3-2|}$.

if $|x| < 1$, then by $\Delta \leq$, $|1+x^2| \leq 1 + |x|^2 \leq 2$.

(Reverse $\Delta \leq$: $|a-b| \geq \big||a|-|b|\big|$)

$|x^3-2| \geq \big||x|^3 - |2|\big| > 1$.

So for $|x| < 1$, $\left| \frac{x+2}{x^3-2} + 1 \right| < 2|x|$.

So take $\delta = \min\{1, \frac{\varepsilon}{2}\}$, then

$\left| \frac{x+2}{x^3-2} + 1 \right| < 2|x| < \varepsilon$. ✓

b) Show that $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ diverges.

Pf: Use sequential criteria. So we need to show $\exists\, x_n, y_n$ s.t. $x_n \to 0$, $y_n \to 0$ as $n \to \infty$, but $\lim_{n\to\infty} \sin\left(\frac{1}{x_n}\right) \neq \lim_{n\to\infty} \sin\left(\frac{1}{y_n}\right)$.

$x_n = \frac{1}{2n\pi} \to 0$ $\qquad$ $\sin\left(\frac{1}{x_n}\right) = \sin(2n\pi) = 0$

$y_n = \frac{1}{2n\pi + \frac{\pi}{2}} \to 0$ by A.P., and $\qquad$ $\sin\left(\frac{1}{y_n}\right) = \sin\left(2n\pi + \frac{\pi}{2}\right) = 1.$

so $\lim_{n\to\infty} \sin\left(\frac{1}{x_n}\right) \neq \lim_{n\to\infty} \sin\left(\frac{1}{y_n}\right)$.

Q2: c) Show that Cauchy sequences are bounded.

By Cauchy, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. if $m, n \geq N$, $|x_m - x_n| < \varepsilon$.

Then by (reversed) triangle inequality, $\big||x_m| - |x_N|\big| < \varepsilon \Rightarrow |x_m| < \varepsilon + |x_N|$ for all $m \geq N$

So for all $m \in \mathbb{N}$, $|x_m| \leq \max\{|x_1|, |x_2|, \dots, |x_N|, \varepsilon + |x_N|\}$.

d) Prove that $\{x_n\}$ converges iff it is Cauchy.

$\Rightarrow$: Since $\{x_n\}$ converges, $\exists x \in \mathbb{R}$ s.t. $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. if $n \geq N$, $|x_n - x| < \frac{\varepsilon}{2}$

So for $m, n \geq N$,

$$|x_m - x_n| = |x_m - x + x - x_n| \overset{\Delta\leq}{\leq} \underbrace{|x_m - x|}_{m \geq N} + |x_n - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

, so $\{x_n\}$ is Cauchy.

$\Leftarrow$: Since $\{x_n\}$ is Cauchy, it is bounded. So by Bolzano-Weierstrass Thm, $\{x_n\}$ admits a convergent subsequence. $\{x_{n_k}\}$ with limit x.

Since $\{x_n\}$ is Cauchy, $\exists N \in \mathbb{N}$ s.t. if $m, n \geq N$, $|x_m - x_n| < \frac{\varepsilon}{2}$.

Since $\{x_{n_k}\}$ is convergent, $\exists K \geq N$ belonging to $\{n_1, n_2, \ldots, n_k, \ldots\}$ s.t.

$|x_K - x| < \frac{\varepsilon}{2}$.

So taking $m = K$, we get for all $n \geq N$,

$$|x_n - x| = |x_n - x_K + x_K - x| \leq |x_n - x_K| + |x_K - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. //$$

Q1: $x_1 = 1$, $x_{n+1} = \sqrt{3x_n}$ for all $n \in \mathbb{N}$.

a) Show by induction that $1 \leq x_n \leq x_{n+1} \leq 3$.

Base Case: $1 = x_1 < \sqrt{3} = x_2 < 3$. ✓

Now sps $1 \leq x_k \leq x_{k+1} \leq 3$ for some k.

$$\Downarrow$$

$$3 \leq 3x_k \leq 3x_{k+1} \leq 9 \Rightarrow 1 < \sqrt{3} \leq \sqrt{3x_k} \leq \sqrt{3x_{k+1}} < 3$$

$$\Rightarrow 1 < \sqrt{3} \leq x_{k+1} \leq x_{k+2} < 3.$$ ✓

b) Converges by MCT.

c) Let $x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1}$. Then $x = \sqrt{3x} \Rightarrow x^2 - 3x = 0 \Rightarrow x = 0, 3$

$\Rightarrow x = 3$ since $x > 0$. ✓